Also solved by E. E. Whitford, E. B. Escott, Elijah Swift, Norman Anning, R. D. Bohannan, George Blanchard, R. E. Gains, and E. F. Canaday.

449. Proposed by FRANK IRWIN, University of California.

Sum the expression

$$1 + 2\binom{k+1}{k} + 3\binom{k+2}{k} + \cdots + (n-k+1)\binom{n}{k}.$$

Also show how to sum

$$1\cdot 2 + 2\cdot 3\binom{k+1}{k} + 3\cdot 4\binom{k+2}{k} + \cdots + (n-k+1)(n-k+2)\binom{n}{k},$$

$$1\cdot 2\cdot 3 + 2\cdot 3\cdot 4\binom{k+1}{k} + 3\cdot 4\cdot 5\binom{k+2}{k} + \cdots + (n-k+1)(n-k+2)(n-k+3)\binom{n}{k},$$

etc., where $\binom{l}{k}$ is used to denote the coefficient of x^k in $(1+x)^l$.

Solution by A. M. Kenyon, Purdue University.

The sum of the first n binomial coefficients in any column of Pascal's triangle is given by the formula,

(1) $$\sum_{i=0}^{n-1}\binom{k+i}{k} = \binom{k+n}{k+1} = \binom{k+n}{n-1}, \qquad \begin{cases} k = 0, 1, 2, \cdots \\ n = 1, 2, 3, \cdots \end{cases}$$

as may be verified and established by induction.

Making use of the notation,

$$x^{(n)} = \prod_{i=0}^{n-1}(x-i), \qquad x^{[n]} = \prod_{i=1}^{n}(x+i), \qquad n = 1, 2, 3, \cdots$$

$$x^{(0)} = x^{[0]} = 1,$$

we have

$$i^{(m)}\binom{k+i}{k} = i^{(m)}\binom{k+i}{i} = k^{[m]}\binom{k+i}{k+m}, \qquad m = 0, 1, 2, \cdots$$

and this in (1) gives,

(2) $$\sum_{i=0}^{n-1} i^{(m)}\binom{k+i}{k} = k^{[m]}\binom{k+n}{k+1+m}, \qquad \begin{cases} k, m = 0, 1, 2, \cdots \\ n = 1, 2, 3, \cdots \end{cases}$$

From the relations[1] among the coefficients of the polynomial in x which result from expanding $x^{(n)}$ we find,

$$i^{[m]} = \sum_{j=0}^{m}\binom{m}{j}^2 \underline{|m-j}\, i^{(j)}, \qquad m = 0, 1, 2, \cdots$$

whence on making use of (2)

(3) $$\sum_{i=0}^{n-1} i^{[m]}\binom{k+i}{k} = \sum_{i=0}^{m}\binom{m}{i}^2 \underline{|m-i}\, k^{[i]}\binom{k+n}{k+1+i}, \qquad \begin{cases} k, m = 0, 1, 2, \cdots \\ n = 1, 2, 3, \cdots \end{cases}$$

Since the problem proposed requires the sum of the first $n-k+1$ terms, we put $n-k+1$ for n,

(4) $$\sum_{i=0}^{n-k} i^{[m]}\binom{k+i}{k} = \sum_{i=0}^{m}\binom{m}{i}^2 \underline{|m-i}\, k^{[i]}\binom{n+1}{k+1+i}, \qquad \begin{cases} m = 0, 1, 2, \cdots \\ n > k = 0, 1, 2, \cdots \end{cases}$$

Setting $m = 0, 1, 2, 3$, etc., in (4), we get

[1] See, "Some Properties of Binomial Coefficients," *Indiana Academy of Science*, 1914, p. 449.

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1};$$

$$1\binom{k}{k} + 2\binom{k+1}{k} + 3\binom{k+2}{k} + \cdots + (n-k+1)\binom{n}{k} = \binom{n+1}{k+1} + (k+1)\binom{n+1}{k+2};$$

$$1\cdot 2\binom{k}{k} + 2\cdot 3\binom{k+1}{k} + 3\cdot 4\binom{k+2}{k} + \cdots + (n-k+1)(n-k+2)\binom{n}{k}$$
$$= 2\binom{n+1}{k+1} + 4k^{[1]}\binom{n+1}{k+2} + k^{[2]}\binom{n+1}{k+3};$$

and

$$1\cdot 2\cdot 3\binom{k}{k} + 2\cdot 3\cdot 4\binom{k+1}{k} + 3\cdot 4\cdot 5\binom{k+2}{k} + \cdots + (n-k+1)(n-k+2)(n-k+3)\binom{n}{k}$$
$$= 6\binom{n+1}{k+1} + 18k^{[1]}\binom{n+1}{k+2} + 9k^{[2]}\binom{n+1}{k+3} + k^{[3]}\binom{n+1}{k+4}.$$

Excellent solutions were also received from E. B. Escott, L. C. Mathewson, Elijah Swift, S. C. Withers, and the Proposer.

GEOMETRY.

A Correction.—Professor R. A. Johnson, Cleveland, Ohio, has called our attention to an error in Professor Clawson's solution of Geometry problem 467, page 50, of the February Monthly.

In lines four and five, Mr. Clawson "refers to two quadrilaterals, using the word in the sense of quadrangle, as being inversely congruent, the corresponding sides being equal and parallel but arranged in opposite orders." Manifestly, two such quadrilaterals cannot be drawn. Mr. Clawson admits the justice of this criticism and says that what he wished to state was the fact that one of the quadrilaterals would have to be turned through two right angles or "inverted" in order to have it similarly placed with the other. Mr. Clawson also says that what he meant by saying in line 5 that the corresponding sides of the quadrilaterals are arranged in opposite order was that the sides are opposite in direction.

Professor Johnson points out that the entire solution can be made rigorous, by deleting the last five words of line 5 and also the word "inversely" wherever it occurs. Editors.

474. Proposed by Laenas G. Weld, Pullman, Ill.

Upon a fixed and constant base stands a system of co-planar triangles, for each of which the radius of the inscribed circle is to that of the circumscribed circle as 1 : 2. What is the locus of the vertices opposite to the given fixed base?

Solution by J. W. Clawson, Collegeville, Pa.

Denote the length of the base by a, the varying sides by r and t, the base angle opposite the side t by θ, radius of incircle by i, and radius of circumcircle by R.

Now

$$R = t/2\sin\theta \text{ and } 2i = (a + r - t)\tan\theta/2.$$

Hence, if $2i = R$, $t = 4(a + r - t)\sin^2\theta/2$, from which we obtain, $t = \dfrac{2(a+r)(1-\cos\theta)}{3 - 2\cos\theta}$.

Again,

$$t^2 = a^2 + r^2 - 2ar\cos\theta.$$